Fixed-Point Error Analysis of Stochastic Gradient Adaptive Lattice Filters

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Abstract—**This paper presents a theoretical analysis of the stochastic gradient adaptive lattice filter used as a linear, one-step predictor, when the effects of finite precision arithmetic are taken into account. Only the fixed-point implementation is considered here. Both the unnormalized and normalized adaptation algorithms are analyzed. Expressions for the steady-state mean-squared values of the accumulated numerical errors in the computation of the reflection coefficients and the prediction errors of different orders have been developed. The results show that the dominant term in the expressions for the mean-squared values of the numerical errors is inversely proportional to the convergence parameter. Furthermore, they indicate that the quantization errors associated with the reflection coefficients are more critical than those associated with representing the prediction error sequences. Another interesting result is that signals with high correlation among samples produce larger numerical errors in the adaptive lattice filter than signals with low correlation among samples. We present several simulation examples that show close agreement with the theoretical results. We also present some comparisons between the numerical behavior of the lattice and transversal stochastic gradient adaptive filters. The numerical results support the general belief that the gradient adaptive lattice filters have better numerical properties than their transversal counterparts, even though it is conceivable that the lattice filters can produce larger numerical errors than the transversal filters under some circumstances.**

I. Introduction

THREE critical considerations in the design of digital adaptive filters are 1) the adaptation algorithm, 2) the filter structure, and 3) the effects of finite precision arithmetic on the filter characteristics. This paper presents a theoretical analysis of the stochastic gradient adaptive lattice filter when the effects of finite precision arithmetic are taken into account. Only fixed-point implementation is considered here. A relatively extensive theoretical analysis establishing the performance of nonadaptive digital filters and their specific hardware realizations can be found in [11]. Caraiscos and Liu [3] have developed expressions for the steady-state mean-squared error of the stochastic gradient transversal adaptive filters when implemented using finite precision arithmetic. Analysis of such filters during adaptation is given in [1]. An empirical study of transversal and lattice filters that use the stochastic gradient algorithm was done in [14]. However, no theoretical analysis of the effect of finite precision arithmetic on the gradient adaptive lattice filter has been available until now. On a related note, Samson and Reddy [13] have studied the numerical properties of a square-root normalized least-square lattice algorithm. Also, infinite precision analyses of gradient adaptive lattice filters can be found in [9] and [16].

Lattice filters, as opposed to their transversal counterparts, have several advantages, and consequently, fixed and adaptive lattice filters have been employed in a variety of applications including noise cancellation [8], channel equalization [15], spectral estimation [6], [18], and linear predictive analysis [12]. Since the lattice filter orthogonalizes the input signals, the gradient adaptation algorithms using this structure are less dependent on the eigenvalue spread of the input signal and may converge faster than their transversal counterparts [9], [15]. The computational complexity for an Nth-order stochastic gradient lattice filter is between $3N$ to $9N$ multiplications per iteration, depending on the specific adaptation algorithm employed. While this complexity is comparable to that of the fast, transversal, recursive least-squares (RLS) algorithms [5] (RLS algorithms have superior convergence properties), their better numerical properties and the reduced dependence of their convergence behavior on the input signal statistics (when compared with gradient transversal filters) make the stochastic gradient lattice adaptive filter a very attractive digital signal processing tool.

Fixed lattice filters are known to have superior numerical properties to that exhibited by transversal filters [4]. While it is possible under some circumstances for the stochastic gradient lattice filters to produce larger numerical errors than transversal adaptive filters, several simulation examples and also numerical comparison of the analytical results have shown that adaptive lattice filters considered in this paper have, in general, better numerical properties than their transversal counterparts.

In this paper, we consider only the lattice filter used as a one-step linear predictor. The extension to the more general filter structure is straightforward. The Nth-order lattice predictor is specified by the recursive equations

$$e_f(n|m) = e_f(n|m-1) - k_m(n)\, e_b(n-1|m-1); \quad m = 1, 2, \cdots, N, \tag{1}$$

and

$$e_b(n|m) = e_b(n-1|m-1) - k_m(n)\, e_f(n|m-1); \quad m = 1, 2, \cdots, N, \tag{2}$$

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where $e_f(n|m)$ and $e_b(n|m)$ are the mth-order forward and backward prediction error sequences, respectively, and $k_m(n)$ is the reflection coefficient at the mth state and time n. The zeroth-order forward and backward prediction errors are given by

$$e_f(n|0) = e_b(n|0) = x(n), \tag{3}$$

where $x(n)$ is the input signal to the predictor. The unnormalized adaptation algorithm studied in this paper updates the reflection coefficients using [6]

$$k_m(n+1) = k_m(n) + \frac{\mu_m}{2}\left\{e_f(n|m)\, e_b(n-1|m-1) + e_b(n|m)\, e_f(n|m-1)\right\}, \tag{4}$$

where μ_m, $m = 1, 2, \cdots, N$ are constants that control the convergence of the adaptive filter. There are several other unnormalized lattice filter update algorithms available in the literature (for example, see [10]), and many of them can be analyzed using techniques similar to that in this paper.

In many applications, it is convenient to use a normalized version of the update algorithm given by (4). In such cases, the reflection coefficients are updated using [8]

$$\begin{aligned} k_m(n+1) &= k_m(n) + \frac{\mu}{\sigma_e(n|m)}\left\{e_f(n|m)\, e_b(n-1|m-1) + e_b(n|m)\, e_f(n|m-1)\right\}; \\ &\quad m = 1, 2, \cdots, N, \end{aligned} \tag{5}$$

where $\sigma_e(n|m)$ is an estimate of the sum of the mean-squared values of the input signals to the mth stage of the lattice predictor at time n, and is given by

$$\sigma_e(n|m) = \beta\sigma_e(n-1|m) + (1-\beta)\left\{e_f^2(n|m-1) + e_b^2(n-1|m-1)\right\}. \tag{6}$$

In the above equations, μ is a constant controlling the convergence of the algorithm and β, $0 < \beta < 1$, is a smoothing parameter used to estimate the signal powers. It is traditional to choose β and μ such that

$$\beta = 1 - \mu. \tag{7}$$

The rest of the paper is organized as follows. Section II presents models for the propagation of the numerical errors in the normalized and unnormalized versions of the stochastic gradient adaptive lattice predictor. Recursive expressions for the steady-state mean-squared values of the numerical errors in the prediction error sequences and reflection coefficients at each stage are summarized in Section III. This section also contains a discussion of the filter parameters and signal characteristics on the numerical properties of the adaptive lattice predictor. Most of the detailed derivations are given in Appendixes A and B. Simulation examples that show close agreement with the theoretical results, and also compare the performance of the lattice predictor with that of the transversal predictor, are presented in Section IV. Finally, Section V contains the concluding remarks.

II. Finite Precision Error Models for the Lattice Predictor

We will assume that the adaptive filter is implemented using fixed-point binary representation with rounding. We will also assume that the input signal has been properly scaled so that all overflow errors are avoided. Thus, one needs to consider only multiplication operations as sources of numerical errors. In what follows, primed variables will denote finite precision terms and unprimed variables will denote their infinite precision counterparts. Thus, using the same notations as in Section I, we have the following relationships between the finite precision and infinite precision variables in the lattice predictor.

$$e_f'(n|m) = e_f(n|m) + \epsilon_f(n|m); \quad m = 0, 1, 2, \cdots, N, \tag{8a}$$

$$e_b'(n|m) = e_b(n|m) + \epsilon_b(n|m); \quad m = 0, 1, 2, \cdots, N, \tag{8b}$$

and

$$k_m'(n) = k_m(n) + \epsilon_k(n|m); \quad m = 1, 2, \cdots, N. \tag{8c}$$

Our objective is to find the steady-state mean-squared values of the numerical errors $\epsilon_f(n|m)$, $\epsilon_b(n|m)$, and $\epsilon_k(n|m)$. The following equations describe the finite precision update of the unnormalized lattice predictor variables:

$$e_f'(n|m) = e_f'(n|m-1) - k_m'(n)\, e_b'(n-1|m-1) + \eta_f(n|m); \quad m = 1, 2, \cdots, N, \tag{9a}$$

$$e_b'(n|m) = e_b'(n-1|m-1) - k_m'(n)\, e_f'(n|m-1) + \eta_b(n|m); \quad m = 1, 2, \cdots, N, \tag{9b}$$

and

$$k_m'(n+1) = k_m'(n) + \frac{\mu_m}{2}\left\{e_f'(n|m)\, e_b'(n-1|m-1) + e_b'(n|m)\, e_f'(n|m-1)\right\} + \eta_k(n+1|m); \quad m = 1, 2, \cdots, N, \tag{9c}$$

where $\eta_f(n|m)$, $\eta_b(n|m)$, and $\eta_k(n|m)$ are the roundoff errors that occur during the computation of $e_f'(n|m)$, $e_b'(n|m)$, and $k_m'(n)$, respectively.

For the normalized lattice filter, the finite precision model can be derived by recognizing that the infinite precision-normalized version can be obtained by replacing $\mu_m/2$ in (4) with $\mu_m(n) = \mu/\sigma_e(n|m)$. The finite precision update equations for the reflection coefficients of the normalized lattice filter are given by

$$k'_m(n+1) = k'_m(n) + \frac{\mu}{\sigma'_e(n|m)} \{e'_f(n|m)\, e'_b(n-1|m-1) + e'_b(n|m)\, e'_f(n|m-1)\} + \eta_{kN}(n+1|m). \quad (10)$$

Also, the finite precision update equation for $\sigma_e(n|m)$ is

$$\sigma'_e(n|m) = \beta\sigma'_e(n-1|m) + (1-\beta)\{e_f'^2(n|m-1) + e_b'^2(n-1|m-1)\} + \eta_\sigma(n|m). \quad (11)$$

Again, $\eta_\sigma(n|m)$ corresponds to the roundoff errors that occur during the computation of $\sigma'_e(n|m)$. Let

$$\sigma'_e(n|m) = \sigma_e(n|m) + \epsilon_\sigma(n|m). \quad (12)$$

Substituting (12) in (10), and using a first-order approximation under the assumption that $|\epsilon_\sigma(n|m)| < \sigma_e(n|m)$, the finite precision update equations for the reflection coefficients can be approximately written as

$$k'_m(n+1) \approx k'_m(n) + \frac{\mu}{\sigma_e(n|m)} \left\{1 - \frac{\epsilon_\sigma(n|m)}{\sigma_e(n|m)}\right\} \cdot \{e'_f(n|m)\, e'_b(n-1|m-1) + e'_b(n|m)\, e'_f(n|m-1)\} + \eta_{kN}(n+1|m) \quad (13a)$$

$$= k'_m(n) + \frac{\mu}{\sigma_e(n|m)} \{e'_f(n|m)\, e'_b(n-1|m-1) + e'_b(n|m)\, e'_f(n|m-1)\} + \xi(n+1|m); \quad m = 1, 2, \cdots, N, \quad (13b)$$

where

$$\xi(n+1|m) = -\frac{\mu}{\sigma_e(n|m)} \frac{\epsilon_\sigma(n|m)}{\sigma_e(n|m)} \{e'_f(n|m)\, e'_b(n-1|m-1) + e'_b(n|m)\, e'_f(n|m-1)\} + \eta_{kN}(n+1|m); \quad m = 1, 2, \cdots, N. \quad (14)$$

The first step toward finding the theoretical steady-state mean-squared values of the numerical errors is to derive the mean-squared values of the various roundoff error terms in (9a)-(14). Before doing that, let us summarize the main assumptions that we will employ for our analysis.

1) The convergence constants μ_m, $m = 1, 2, \cdots, N$ (for the unnormalized case) and the convergence parameter μ (for the normalized case) are small enough to guarantee convergence of the adaptive filter for stationary input signals. Moreover, the mean values of the infinite precision reflection coefficients converge to the optimal values, i.e.,

$$\lim_{n\to\infty} E\{k_m(n)\} = k_{\mathrm{opt}}(m); \qquad m = 1, 2, \cdots, N. \quad (15)$$

Also, μ_m for the unnormalized case, and μ and β for the normalized case, are such that they can be represented exactly. This implies that there is no finite precision error associated with representation of parameters of the lattice filter, and consequently the analysis will be that much simpler.

2) The input signal $x(n)$ belongs to a zero mean and stationary Gaussian process with variance σ_x^2. The steady-state prediction errors at each stage are also realizations of zero mean Gaussian processes. Furthermore, the steady-state reflection coefficients are uncorrelated with the steady-state prediction error sequences. Obviously, when the data are correlated, this assumption will not hold. However, for small values of the convergence parameters, this assumption will produce analytical results that closely match the behavior of practical systems, and therefore has been applied in several analyses of gradient adaptive lattice filters in the past [9], [16]. A recent analysis of a single coefficient gradient adaptive filter [2] has shown that the analysis using correlated data assumption differs from that using uncorrelated data assumption significantly only when the input data are very highly correlated (correlation coefficient greater than 0.99).

3) The roundoff noise sequences $\eta_f(n|m)$, $\eta_b(n|m)$, and $\eta_k(n|m)$ (also $\eta_\sigma(n|m)$ and $\eta_{kN}(n|m)$ when the normalized adaptation is considered) are independent of all data sequences and also with each other. They all belong to zero mean random processes that are uniformly distributed in appropriate ranges. If the data samples and the filter coefficients are represented using b_d and b_c bits, respectively (including the sign bit), then

$$-2^{-b_d} \le \eta_f(n|m),\ \eta_b(n|m) \le 2^{-b_d}. \quad (16)$$

It is easy to see that

$$\sigma_f^2 = E\{\eta_f^2(n|m)\} = E\{\eta_b^2(n|m)\} = \frac{2^{-2(b_d-1)}}{12}; \quad m = 0, 1, 2, \cdots, N. \quad (17)$$

For the unnormalized case, we will assume that the reflection coefficients are updated by implementing the two multiplications within the brackets in (9c), adding up the products and then multiplying this sum by $\mu_m/2$. Then, $\eta_k(n|m)$ will have a variance given by

$$\sigma_k^2(m) = E\{\eta_k^2(n|m)\} = \left(1 + \frac{\mu_m^2}{2}\right) \frac{2^{-2(b_c-1)}}{12}. \quad (18)$$

4) The accumulated noise sequences $\epsilon_f(n|m)$, $\epsilon_b(n|m)$, $\epsilon_\sigma(n|m)$, and $\epsilon_k(n|m)$ are uncorrelated with each other for all values of m and n. Obviously, this is never exactly true, but several simulations have demonstrated that the contribution of statistical expectations of cross-error terms (for example, $E\{\epsilon_k(n|m_1)\,\epsilon_f(n|m_2)\}$, $E\{\epsilon_f(n_1|m_1)\,\epsilon_b(n_2|m_2)\}$, etc.) to the steady-state mean-

squared values of the numerical errors is negligible when compared to the contributions of the variance terms. Furthermore, we will assume that third- and higher-order products of the error terms are negligibly small.

5) For the normalized case, $\sigma_e(n|m)$ is uncorrelated with the other data signals and also the reflection coefficients. This is also never true in practice, but is a good approximation if $1-\beta << 1$ and simulations have shown that the approximation holds under this condition. Also, we will use the following approximate result, again assuming that $1-\beta << 1$:

$$E\left\{\frac{\mu\epsilon_{\sigma}^{j}(n|m)}{(\sigma_e(n|m))^{j}}\right\} \approx \frac{\mu E\{\epsilon_{\sigma}^{j}(n|m)\}}{E^{j}\{\sigma_e(n|m)\}}. \tag{19}$$

We are essentially assuming that the statistical variations of $\sigma_e(n|m)$ are negligible. The above approximation is based on what is known as the averaging principle which has been used successfully in several other situations [9], [13].

III. Summary of Analytical Results

For notational convenience, we will suppress the time index n whenever we deal with steady-state quantities and this suppression of the time index causes no confusion. For example,

$$E\{\epsilon_f^2(m)\} = \lim_{n\to\infty} E\{\epsilon_f^2(n|m)\}. \tag{20}$$

Also, note that, because of the symmetry of the problems (both the normalized and unnormalized cases),

$$E\{\epsilon_f^2(m)\} = E\{\epsilon_b^2(m)\}; \qquad m = 0, 1, 2, \cdots, N, \tag{21}$$

and

$$E\{e_f^2(m)\} = E\{e_b^2(m)\}; \qquad m = 0, 1, 2, \cdots, N. \tag{22}$$

A. Unnormalized Case

Using the infinite precision update equations given by (1), (2), and (4), the corresponding finite precision update equations (9a)–(9c), and the definitions of the numerical errors in (8a)–(8c), we can derive the following set of recursive equations for the steady-state mean-squared numerical errors. The derivations are given in Appendix A.

$$E\{k_m^2\} = \frac{k_{\text{opt}}^2(m) + \frac{\mu_m}{2}E\{e_f^2(m-1)\}(1-4k_{\text{opt}}^2(m))}{1-\frac{\mu_m}{2}E\{e_f^2(m-1)\}(2+k_{\text{opt}}^2(m))}, \tag{23}$$

$$E\{\epsilon_k^2(m)\} = \frac{\sigma_k^2(m) + \mu_m^2 E\{e_f^2(m-1)\}\left\{\frac{\sigma_f^2}{2} + 2E\{\epsilon_f^2(m-1)\}(1+E\{k_m^2\}-2k_{\text{opt}}^2(m))\right\}}{\mu_m E\{e_f^2(m-1)\}\{2-\mu_m E\{e_f^2(m-1)\}(2+k_{\text{opt}}^2(m))\}}, \tag{24}$$

$$E\{\epsilon_f^2(m)\} = E\{\epsilon_f^2(m-1)\}(1+E\{k_m^2\}) + E\{\epsilon_k^2(m)\}E\{e_f^2(\text{m}-1)\} + \sigma_f^2, \tag{25}$$

and

$$E\{e_f^2(m)\} = E\{e_f^2(m-1)\}(1+E\{k_m^2\} - 2k_{\text{opt}}^2(m)). \tag{26}$$

Equations (23)–(26) are evaluated recursively for $m = 1, 2, \cdots, N$. To initialize the recursion, note that

$$E\{e_f^2(0)\} = E\{x^2(n)\} = \sigma_x^2, \tag{27}$$

and

$$E\{\epsilon_f^2(0)\} \approx E\{(x(n)-x'(n))^2\} = \sigma_f^2 \tag{28}$$

is the mean-squared quantization error of the input signal $x(n)$.

Equations (23)–(26) recursively generate steady-state mean-squared values for the finite and infinite precision quantities associated with the reflection coefficients and the prediction error sequences. To get a better feel for the numerical properties, we will consider the case for which the convergence constants μ_m's are small enough to neglect the variability of the steady-state quantities due to adaptation. Then from (23) and (26)

$$E\{k_m^2\} \approx k_{\text{opt}}^2(m), \tag{29}$$

and

$$E\{e_f^2(m)\} \approx E\{e_f^2(m-1)\}(1-k_{\text{opt}}^2(m)); \quad m = 1, 2, \cdots, N. \tag{30}$$

Using the same approximation (that μ_m is small), we get the following expression for $E\{\epsilon_k^2(m)\}$ from (24):

$$E\{\epsilon_k^2(m)\} \approx \frac{\sigma_k^2(m)}{2\mu_m E\{e_f^2(m-1)\}} + \frac{\mu_m}{2}\left\{\frac{\sigma_f^2}{2} + 2E\{\epsilon_f^2(m-1)\}(1-k_{\text{opt}}^2(m))\right\}. \tag{31}$$

Several remarks are in order here.

Note that for small values of μ_m, the first term dominates the second term on the right-hand side except when $E\{\epsilon_f^2(m-1)\}$ is very large. The dominant terms become very large as μ_m goes to zero. Thus, the adaptive lattice filter, similar to the adaptive transversal filter, produces larger numerical errors with decreasing values of the convergence parameters.

The dominant term depends on the variance of the quantization error in the coefficient update equation (and not on the quantization errors associated with the prediction error sequences). This shows that the filter is more sensitive to errors in representing the reflection coefficients than the prediction error sequences.

Finally, note that the dominant term is inversely proportional to the steady-state mean-squared prediction error values. Thus, the numerical inaccuracies associated with the mth reflection coefficient will be very large for very small values of $E\{e_f^2(m-1)\}$. In other words, the lattice filter will exhibit poor numerical properties if the "predictability" of the input signal is very high. One consequence of this result is that the lattice filter must be implemented with a high degree of precision when working with highly correlated or sinusoidal signals. Another aspect that we should consider is the fact that the prediction error power decreases with increasing prediction order. Therefore, it may be necessary to represent the lattice filter coefficients at later stages with higher precision than those at earlier stages.

Substituting (29) and (31) in (25), we get

$$\begin{aligned} E\{\epsilon_f^2(m)\} &\approx E\{\epsilon_f^2(m-1)\}\,(1+k_{\text{opt}}^2(m)) + \frac{\sigma_k^2(m)}{2\mu_m} \\ &\quad + \frac{\mu_m}{2} E\{e_f^2(m-1)\}\left\{\frac{\sigma_f^2}{2} + 2E\{\epsilon_f^2(m-1)\} \cdot (1-k_{\text{opt}}^2(m))\right\} + \sigma_f^2 \qquad (32) \\ &\approx \frac{\sigma_k^2(m)}{2\mu_m} + E\{\epsilon_f^2(m-1)\}\,(1+k_{\text{opt}}^2(m)) \\ &\quad + \frac{\mu_m}{4}\sigma_f^2 E\{e_f^2(m-1)\} + \sigma_f^2. \qquad (33) \end{aligned}$$

In (33) the second term dominates the term that has been neglected when μ_m is small. Once again, we find a term that is inversely proportional to the convergence parameter in the approximate expression for $E\{\epsilon_f^2(m)\}$. The second term on the right-hand side of (33) is also very interesting. Since $1+k_{\text{opt}}^2$ is always greater than or equal to one (with equality if and only if $k_{\text{opt}}(m)=0$), we see that numerical errors at the previous stages are amplified and propagated into the next stage of the lattice. This implies that the numerical errors may become extremely large when the order of the filter is very large. This also indicates that we can conceive of adaptive lattice filter structures whose numerical properties may be worse than their transversal counterparts. However, our experiments do demonstrate that for several types of signals and parameters, adaptive lattice filters do have better numerical properties than transversal filters, and thus support the general belief that gradient adaptive lattice filters perform better than their transversal counterparts when implemented with finite precision.

B. Normalized Case

Comparing the finite precision update equations for the normalized and unnormalized lattice algorithms [see (9c) and (13)], we find that the only differences between the two are 1) $\mu_m/2$ in the unnormalized version is replaced by $\mu/\sigma_e(n|m)$ and 2) the error term $\eta_k(n+1|m)$ in (9c) is replaced by a different error term $\xi(n+1|m)$ in (13) for the normalized case. Thus, the functional form of the recursive expressions for the steady-state mean-squared values of the accumulated numerical errors will remain the same as (23)–(26), except that $\mu_m/2$ must be replaced by (see Appendix B)

$$E\left\{\frac{\mu}{\sigma_e(n|m)}\right\} \approx \frac{\mu/2}{E\{e_f^2(m-1)\}} \qquad (34)$$

and $\sigma_k^2(m)$ must be replaced by $E\{\xi^2(m)\}$ as obtained in (B8). Note that these results make use of the assumption that $\sigma_e(n|m)$ is uncorrelated with the prediction errors and reflection coefficients.

IV. Simulation Examples

In this section, we present the results of simulation examples that demonstrate the validity of the results derived in this paper. We used two sets of signals for most of our experiments. The first one (TS1) was a third-order autoregressive signal obtained by processing a zero mean, white Gaussian pseudorandom noise sequence with unit variance with a third-order all-pole filter with transfer function

$$H_1(z) = \frac{0.25}{(1-0.8z^{-1})(1-0.4z^{-1})(1-0.2z^{-1})}. \qquad (35)$$

The second test signal, designated TS2, was obtained by processing a zero mean, white Gaussian pseudorandom sequence with unit variance with a seventh-order all-pole filter with transfer function

$$H_2(z) = \frac{0.4}{(1-0.3z^{-1})^2(1+z^{-1}+0.5z^{-2})(1-0.4z^{-1})(1-0.5z^{-2})}. \qquad (36)$$

In experiments involving TS1, the adaptive filters were run as 3rd-order predictors. When TS2 was used, the filters were run as 7th-order predictors. The results presented are all steady-state values. Theoretical results were obtained using the recursive expressions developed in the paper. The simulations were done on a Gould 9080 computer, and the numerical errors were computed by running two parallel filters—one that utilizes the maximum precision available in the system, and the other with a given

TABLE I
$E\{\epsilon_f^2(3)\}$ FOR THE THIRD-ORDER TEST EXAMPLE AND UNNORMALIZED UPDATE

μ	b_c \ b_d	16 Theor.	16 Simul.	12 Theor.	12 Simul.	8 Theor.	8 Simul.
2^{-2}	16	0.11×10^{-8}	0.12×10^{-8}	0.12×10^{-6}	0.13×10^{-6}	0.29×10^{-4}	0.32×10^{-4}
	12	0.16×10^{-6}	0.18×10^{-6}	0.27×10^{-6}	0.30×10^{-6}	0.29×10^{-4}	0.32×10^{-4}
	8	0.40×10^{-4}	0.39×10^{-4}	0.40×10^{-4}	0.41×10^{-4}	0.70×10^{-4}	0.68×10^{-4}
2^{-3}	16	0.15×10^{-8}	0.18×10^{-8}	0.11×10^{-6}	0.10×10^{-6}	0.29×10^{-4}	0.33×10^{-4}
	12	0.28×10^{-6}	0.26×10^{-6}	0.39×10^{-6}	0.36×10^{-6}	0.29×10^{-4}	0.33×10^{-4}
	8	0.71×10^{-4}	0.76×10^{-4}	0.71×10^{-4}	0.73×10^{-4}	0.10×10^{-3}	0.12×10^{-3}
2^{-4}	16	0.25×10^{-8}	0.28×10^{-8}	0.11×10^{-6}	0.11×10^{-6}	0.29×10^{-4}	0.33×10^{-4}
	12	0.53×10^{-6}	0.51×10^{-6}	0.64×10^{-6}	0.63×10^{-6}	0.29×10^{-4}	0.33×10^{-4}
	8	0.14×10^{-3}	0.12×10^{-3}	0.14×10^{-3}	0.16×10^{-3}	0.16×10^{-3}	0.16×10^{-3}

TABLE II
$E\{\epsilon_f^2(7)\}$ FOR THE SEVENTH-ORDER TEST EXAMPLE AND UNNORMALIZED UPDATE

μ	b_c \ b_d	16 Theor.	16 Simul.	12 Theor.	12 Simul.	8 Theor.	8 Simul.
2^{-2}	16	0.26×10^{-8}	0.23×10^{-8}	0.24×10^{-6}	0.24×10^{-6}	0.52×10^{-4}	0.51×10^{-4}
	12	0.42×10^{-6}	0.39×10^{-6}	0.66×10^{-6}	0.63×10^{-6}	0.52×10^{-4}	0.51×10^{-4}
	8	0.11×10^{-3}	0.10×10^{-3}	0.11×10^{-3}	0.10×10^{-3}	0.17×10^{-3}	0.16×10^{-3}
2^{-3}	16	0.34×10^{-8}	0.36×10^{-8}	0.21×10^{-6}	0.22×10^{-6}	0.51×10^{-4}	0.51×10^{-4}
	12	0.67×10^{-6}	0.65×10^{-6}	0.88×10^{-6}	0.89×10^{-6}	0.52×10^{-4}	0.51×10^{-4}
	8	0.17×10^{-3}	0.17×10^{-3}	0.17×10^{-3}	0.16×10^{-3}	0.22×10^{-3}	0.23×10^{-3}
2^{-4}	16	0.56×10^{-8}	0.59×10^{-8}	0.20×10^{-6}	0.20×10^{-6}	0.52×10^{-4}	0.50×10^{-4}
	12	0.12×10^{-5}	0.11×10^{-5}	0.14×10^{-5}	0.15×10^{-5}	0.53×10^{-4}	0.52×10^{-4}
	8	0.32×10^{-3}	0.30×10^{-3}	0.32×10^{-3}	0.31×10^{-3}	0.36×10^{-3}	0.34×10^{-3}

numerical precision. All the empirical steady-state results are obtained as the time averages over the last 200 iterations of the ensemble averages over 100 independent runs using 2000 data samples each.

Tables I–IV display matrices obtained by comparing the theoretical and empirical steady-state mean-squared numerical errors accumulated in computing the forward prediction error (3rd-order for TS1 and 7th-order for TS2). Both the normalized and unnormalized versions are considered here. Tests were run using several values of μ (for the unnormalized filter we selected $\mu_m = \mu$ for all m), and b_c and b_d. When the normalized lattice filter was employed, b_μ was set to be the same as b_c and β was selected as $\beta = 1 - \mu$. We can make several observations at this point.

1) The theoretical and empirical results show very good match. The small differences between the analytical and simulation results can be attributed to the approximations employed and also the statistical variability of the experiments.

2) Both analysis [see (24), (25), (31), and (32)] and simulations indicate that the numerical errors increase with decreasing values of the convergence parameters. This result is consistent with a similar result obtained for the transversal filters in [3].

3) A comparison of the mean-squared numerical errors indicates that it is important to represent the coefficients of the filter with higher precision than the data themselves. Once again, this finding agrees with similar results for the transversal filters [1], [3] and the discussion in Section III.

4) If we use an adequate number of bits to represent the data and coefficients, the excess mean-squared estimation error due to finite precision implementation may be negligible when compared to the excess mean-squared estimation error due to adaptation. If only a small number of bits are used to represent the data and the filter coefficients, then there may be an optimum value of the convergence parameter that minimizes the total excess mean-squared error due to adaptation and finite precision implementation.

5) It is generally believed that the stochastic gradient lattice adaptive filters have better numerical properties than the transversal, gradient adaptive filters. The discussion in Section III indicates that at least for very large order systems, this may not be the case. In spite of this,

TABLE III
$E\{\epsilon_f^2(3)\}$ FOR THE THIRD-ORDER TEST EXAMPLE AND NORMALIZED UPDATE ($b_c = b_\mu$)

μ	b_c \ b_d	16 Theor.	16 Simul.	14 Theor.	14 Simul.	12 Theor.	12 Simul.
2^{-6}	16	0.33×10^{-8}	0.38×10^{-8}	0.96×10^{-8}	0.11×10^{-7}	0.12×10^{-6}	0.13×10^{-6}
	14	0.57×10^{-7}	0.54×10^{-7}	0.63×10^{-7}	0.61×10^{-7}	0.15×10^{-6}	0.18×10^{-6}
	12	0.93×10^{-6}	0.10×10^{-5}	0.10×10^{-5}	0.11×10^{-5}	0.94×10^{-6}	0.12×10^{-5}
2^{-7}	16	0.67×10^{-8}	0.75×10^{-8}	0.12×10^{-7}	0.15×10^{-7}	0.12×10^{-6}	0.14×10^{-6}
	14	0.15×10^{-6}	0.10×10^{-6}	0.97×10^{-7}	0.11×10^{-6}	0.19×10^{-6}	0.23×10^{-6}
	12	0.38×10^{-5}	0.40×10^{-5}	0.47×10^{-5}	0.40×10^{-5}	0.48×10^{-5}	0.42×10^{-5}
2^{-8}	16	0.96×10^{-8}	0.11×10^{-7}	0.23×10^{-7}	0.19×10^{-7}	0.14×10^{-6}	0.14×10^{-6}
	14	0.36×10^{-6}	0.34×10^{-6}	0.31×10^{-6}	0.34×10^{-6}	0.51×10^{-6}	0.49×10^{-6}
	12	0.88×10^{-5}	0.94×10^{-5}	0.91×10^{-5}	0.97×10^{-5}	0.93×10^{-5}	0.98×10^{-5}

TABLE IV
$E\{\epsilon_f^2(7)\}$ FOR THE SEVENTH-ORDER TEST EXAMPLE AND NORMALIZED UPDATE ($b_c = b_\mu$)

μ	b_c \ b_d	18 Theor.	18 Simul.	14 Theor.	14 Simul.	10 Theor.	10 Simul.
2^{-4}	18	0.22×10^{-9}	0.21×10^{-9}	0.17×10^{-7}	0.17×10^{-7}	0.43×10^{-5}	0.42×10^{-5}
	14	0.39×10^{-7}	0.36×10^{-7}	0.56×10^{-7}	0.50×10^{-7}	0.43×10^{-5}	0.42×10^{-5}
	10	0.10×10^{-4}	0.98×10^{-5}	0.10×10^{-4}	0.98×10^{-5}	0.14×10^{-4}	0.13×10^{-4}
2^{-5}	18	0.26×10^{-9}	0.27×10^{-9}	0.14×10^{-7}	0.14×10^{-7}	0.35×10^{-5}	0.35×10^{-5}
	14	0.53×10^{-7}	0.49×10^{-7}	0.66×10^{-7}	0.67×10^{-7}	0.35×10^{-5}	0.35×10^{-5}
	10	0.14×10^{-4}	0.14×10^{-4}	0.14×10^{-4}	0.14×10^{-4}	0.17×10^{-4}	0.17×10^{-4}
2^{-6}	18	0.39×10^{-9}	0.39×10^{-9}	0.13×10^{-7}	0.13×10^{-7}	0.32×10^{-5}	0.33×10^{-5}
	14	0.87×10^{-7}	0.90×10^{-7}	0.99×10^{-7}	0.10×10^{-6}	0.32×10^{-5}	0.33×10^{-5}
	10	0.23×10^{-4}	0.26×10^{-4}	0.23×10^{-4}	0.26×10^{-4}	0.25×10^{-4}	0.27×10^{-4}

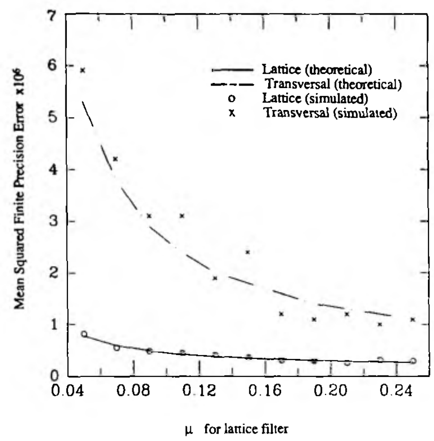

Fig. 1. Comparison of steady-state numerical errors for lattice and transversal filters under the same excess mean-squared error due to adaptation (3rd-order test example unnormalized update with $b_c = b_d = 12$ bits).

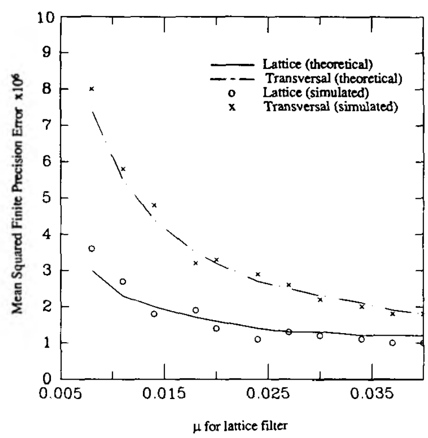

Fig. 2. Comparison of steady-state numerical errors for lattice and transversal filters under the same excess mean-squared error due to adaptation (7th-order test example normalized update with $b_c = b_d = 12$ bits).

it is our belief that the adaptive lattice filters considered here do perform better than their transversal counterparts in a very large number of situations. We present two numerical comparisons here when the filter orders are relatively small. In the results presented in Figs. 1 and 2, the convergence parameters μ_L ($= \mu$ for all m for the unnormalized case) for the lattice filters and μ_T for the corresponding transversal filters were chosen such that the the-

TABLE V
STEADY-STATE MEAN-SQUARED FINITE PRECISION ERRORS AT DIFFERENT STAGES FOR THE SEVENTH-ORDER TEST EXAMPLE

b_c b_d b_μ / μ / FPE	$b_c=14$ $b_d=12$ $\mu=2^{-3}$ Unnorm. Update		$b_c=16$ $b_d=14$ $\mu=2^{-4}$ Unnorm. Update		$b_c=b_\mu=16$ $b_d=14$ $\mu=2^{-5}$ Norm. Update		$b_c=b_\mu=18$ $b_d=16$ $\mu=2^{-6}$ Norm. Update	
	Theor.	Simul.	Theor.	Simul.	Theor.	Simul.	Theor.	Simul.
$E\{\epsilon_f^2(1)\}$	0.62×10^{-7}	0.61×10^{-7}	0.41×10^{-8}	0.41×10^{-8}	0.45×10^{-8}	0.44×10^{-8}	0.34×10^{-9}	0.32×10^{-9}
$E\{\epsilon_f^2(2)\}$	0.90×10^{-7}	0.98×10^{-7}	0.62×10^{-8}	0.67×10^{-8}	0.64×10^{-8}	0.68×10^{-8}	0.47×10^{-9}	0.49×10^{-9}
$E\{\epsilon_f^2(3)\}$	0.12×10^{-6}	0.13×10^{-6}	0.82×10^{-8}	0.88×10^{-8}	0.83×10^{-8}	0.89×10^{-8}	0.60×10^{-9}	0.65×10^{-9}
$E\{\epsilon_f^2(4)\}$	0.15×10^{-6}	0.16×10^{-6}	0.10×10^{-7}	0.11×10^{-7}	0.10×10^{-7}	0.11×10^{-7}	0.74×10^{-9}	0.80×10^{-9}
$E\{\epsilon_f^2(5)\}$	0.18×10^{-6}	0.20×10^{-6}	0.13×10^{-7}	0.14×10^{-7}	0.13×10^{-7}	0.13×10^{-7}	0.88×10^{-9}	0.91×10^{-9}
$E\{\epsilon_f^2(6)\}$	0.22×10^{-6}	0.23×10^{-6}	0.15×10^{-7}	0.16×10^{-7}	0.15×10^{-7}	0.15×10^{-7}	0.10×10^{-8}	0.10×10^{-8}
$E\{\epsilon_f^2(7)\}$	0.25×10^{-6}	0.26×10^{-6}	0.17×10^{-7}	0.18×10^{-7}	0.17×10^{-7}	0.18×10^{-7}	0.12×10^{-8}	0.11×10^{-8}

TABLE VI
$E\{\epsilon_k^2(2)\}$ FOR THREE SIGNALS WITH DIFFERENT PREDICTABILITIES: A) UNPREDICTABLE; B) MODERATELY PREDICTABLE; C) HIGHLY PREDICTABLE

b_c b_d b_μ / μ / Signal	$b_c=12$ $b_d=10$ $\mu=2^{-5}$ Unnorm. Update		$b_c=14$ $b_d=12$ $\mu=2^{-6}$ Unnorm. Update		$b_c=b_\mu=16$ $b_d=14$ $\mu=2^{-7}$ Norm. Update		$b_c=b_\mu=18$ $b_d=16$ $\mu=2^{-8}$ Norm. Update	
	Theor.	Simul.	Theor.	Simul.	Theor.	Simul.	Theor.	Simul.
Unpredictable	0.35×10^{-6}	0.32×10^{-6}	0.41×10^{-7}	0.45×10^{-7}	0.37×10^{-8}	0.44×10^{-8}	0.57×10^{-9}	0.63×10^{-9}
Mod. predictable	0.63×10^{-6}	0.67×10^{-6}	0.85×10^{-7}	0.80×10^{-7}	0.82×10^{-8}	0.78×10^{-8}	0.16×10^{-8}	0.11×10^{-8}
Hi. predictable	0.10×10^{-3}	0.11×10^{-3}	0.21×10^{-5}	0.19×10^{-5}	0.15×10^{-6}	0.13×10^{-6}	0.13×10^{-6}	0.14×10^{-6}

oretical excess mean-squared estimation error due to adaptation were the same for both cases. The theoretical values for the transversal filters were obtained from [3]. We can see that in both the comparisons, the numerical errors for the lattice filters are smaller than that for the transversal filters. While this is no conclusive proof that this is always the case, these results do support the general belief that gradient lattice filters have superior numerical properties than the gradient transversal filters, at least when the filter orders are relatively small.

The analytical and empirical values of the steady-state mean-squared numerical errors in computing the forward prediction errors at different stages ($m = 1, 2, \cdots, 7$) are displayed in Table V when test signal TS2 was used in the experiments. In addition to showing excellent match between theoretical and simulation results, we can see that the numerical errors increase with increasing order of prediction. This result also agrees with our discussion in Section III.

The final set of experiments demonstrates the influence of signal correlation on the numerical behavior of the adaptive lattice filter. We consider a second-order lattice predictor for this set of experiments. Three types of input signals were used in the experiments: zero mean, white Gaussian signal with unit variance, and two autoregressive signals (one with moderately low-pass characteristics and the other with highly low-pass characteristics). Both the autoregressive signals had zero mean value and unit variance. They were obtained by processing zero mean, white Gaussian signal with second-order all-pole filters with transfer functions

$$H_1(z) = \frac{0.65}{1 - 0.99z^{-1} + 0.49z^{-2}}, \tag{37}$$

and

$$H_2(z) = \frac{0.061}{1 - 1.8z^{-1} + 0.81z^{-2}}, \tag{38}$$

respectively. Obviously, the white signal is totally unpredictable and the highly low-pass signal has high correlation between adjacent samples and hence high predictability. The other signal has only "moderate" predictability. Table VI compares the numerical errors in computing the reflection coefficients for the three different signals. The results demonstrate the validity of the discussion in Section III—the higher the predictability, the larger the numerical errors.

V. CONCLUSIONS

In this paper, we presented a theoretical analysis of the numerical properties of stochastic gradient adaptive lattice filters when fixed-point binary arithmetic is used in their implementations. Expressions for the steady-state mean-squared values of the accumulated numerical errors during the computation of the reflection coefficients and the prediction errors were derived. The results show that the dominant term in the expressions for the mean-squared values of the numerical errors is inversely proportional to the convergence parameter. Furthermore, they indicate that the quantization errors associated with the reflection coefficients are more critical than those associated with representing the prediction error sequences. Another interesting result is that signals with high correlation among samples produce larger numerical errors in the adaptive lattice filter than signals with low correlation among samples. Results of some simulation experiments that showed good correlation with analytic results were also presented.

Analysis of several other adaptation algorithms can be done using techniques similar to those presented here. The analysis can also be easily extended to the transient case, even though we did not consider it in our paper.

Appendix A
Derivation of (23)–(26) for the Unnormalized Lattice Predictor

Substituting (1), (8b), and (9a) in (8a), and neglecting products of uncorrelated error terms, the accumulated numerical error sequence $\epsilon_f(n|m)$ for the forward prediction error at the mth stage can be written as

$$\begin{aligned}\epsilon_f(n|m) &= \epsilon_f(n|m-1) - k_m(n)\,\epsilon_b(n-1|m-1) \\ &\quad - e_b(n-1|m-1)\,\epsilon_k(n|m) + \eta_f(n|m); \\ &\qquad m = 1, 2, \cdots, N. \qquad (\text{A1})\end{aligned}$$

Similarly, by combining (1), (2), (4), (8c), and (9c), we get

$$\begin{aligned}&\epsilon_k(n+1|m) \\ &= \epsilon_k(n|m)\left\{1 - \frac{\mu_m}{2}\left(e_f^2(n|m-1) + e_b^2(n-1|m-1)\right)\right\} \\ &\quad - \mu_m k_m(n)\left\{\epsilon_f(n|m-1)\,e_f(n-1|m-1) + \epsilon_b(n-1|m-1)\,e_b(n-1|m-1)\right\} \\ &\quad + \mu_m\left\{\epsilon_f(n|m-1)\,e_b(n-1|m-1) + \epsilon_b(n-1|m-1)\,e_f(n|m-1)\right\} \\ &\quad + \frac{\mu_m}{2}\left\{\eta_f(n|m)\,e_b(n-1|m-1) + \eta_b(n|m)\,e_f(n|m-1)\right\} + \eta_k(n+1|m); \\ &\qquad m = 1, 2, \cdots, N. \qquad (\text{A2})\end{aligned}$$

The recursive relationship for $E\{\epsilon_f^2(m)\}$ given by (25) can be derived by squaring and taking the statistical expectations of both sides of (A1) as n goes to ∞ and then simplifying using assumptions 3) and 4) of Section II and also the equalities given by (21) and (22).

Squaring both sides of (A2) and taking the statistical expectations as n goes to ∞ after ignoring the contributions due to the product of noise sequences that are assumed to be uncorrelated, we get the following expression:

$$\begin{aligned}E\{\epsilon_k^2(m)\} &= E\{\epsilon_k^2(m)\}\left\{1 - 2\mu_m E\{e_f^2(m-1)\} + \mu_m^2 E^2\{e_f^2(m-1)\}\left(2 + k_{\text{opt}}^2(m)\right)\right\} \\ &\quad + \sigma_k^2(m) + \mu_m^2 E\{e_f^2(m-1)\} \\ &\quad \cdot \left\{\frac{\sigma_f^2}{2} + 2E\{\epsilon_f^2(m-1)\} \cdot \left(1 + E\{k_m^2\} - 2k_{\text{opt}}^2(m)\right)\right\}. \qquad (\text{A3})\end{aligned}$$

In deriving (A3), we made use of the fact that the infinite precision variables $e_f(n|m)$ and $e_b(n|m)$ belong to zero mean Gaussian processes and expressed several fourth-order expectations in terms of second-order statistical expectations using the result [15].

$$\begin{aligned}E\{X_1X_2X_3X_4\} &= E\{X_1X_2\}\,E\{X_3X_4\} + E\{X_1X_3\}\,E\{X_2X_4\} \\ &\quad + E\{X_1X_4\}\,E\{X_2X_3\}. \qquad (\text{A4})\end{aligned}$$

Also, we made use of the approximation that

$$\begin{aligned}&\lim_{n\to\infty} E\{e_f(n|m-1)\,e_b(n-1|m-1)\} \\ &\quad \approx k_{\text{opt}} E\{e_f^2(m-1)\}. \qquad (\text{A5})\end{aligned}$$

Equation (24) follows immediately from (A3). To complete the analysis, we need expressions for the steady-state mean-squared values of the infinite precision quantities $e_f(n|m)$ and $k_m(n)$. Equation (26) can be easily derived by squaring and taking the statistical expectation of both sides of (1) as n goes to ∞ and then simplifying using the uncorrelatedness of the reflection coefficients with the prediction error sequences in the steady state, the equality in (22), and also the approximation in (A4).

Finally, to obtain $E\{k_m^2\}$, we can proceed as before. Combining (4), (1), and (2) and simplifying, we get

$$\begin{aligned}k_m(n+1) &= k_m(n)\left\{1 - \frac{\mu_m}{2}\left(e_b^2(n-1|m-1) + e_f^2(n|m-1)\right)\right\} + \mu_m e_f(n|m-1) \\ &\quad \cdot e_b(n-1|m-1). \qquad (\text{A6})\end{aligned}$$

The steady-state mean-squared value of $k_m(n)$ can be obtained by squaring both sides of (A6), taking the statistical expectations of both sides of the resulting expression as n goes to ∞ and then simplifying using the assumptions 1) and 2) given in Section II, and also using the equalities in (21) and (22). This gives

$$\begin{aligned}E\{k_m^2\} &= E\{k_m^2\}\left\{1 + \mu_m^2 E^2\{e_f^2(m-1)\} \cdot \left(2 + k_{\text{opt}}^2(m)\right) - 2\mu_m E\{e_f^2(m-1)\}\right\} \\ &\quad + 2\mu_m^2 k_{\text{opt}}^2(m)\,E\{e_f^2(m-1)\} \\ &\quad + \mu_m^2 E^2\{e_f^2(m-1)\}\left(1 - 4k_{\text{opt}}^2(m)\right); \\ &\qquad m = 1, 2, \cdots, N. \qquad (\text{A7})\end{aligned}$$

Equation (23) results from solving for $E\{k_m^2\}$ from (A7).

Appendix B
Evaluation of $E\{\xi^2(m)\}$

Assume that $\sigma_e(n|m)$ is represented using b_μ bits, including the sign bit. Suppose that the update equation (11) for $\sigma_e(n|m)$ is implemented by first squaring $e_f'(n|m-1)$ and $e_b'(n|m-1)$, summing them, multiplying the sum with $(1-\beta)$, and then adding to it $\beta\sigma_e(n|m)$. Then, we can show as in (18) that

$$\sigma_\eta^2 = E\{\eta_\sigma^2(n|m)\} = 2[1+(1-\beta)^2]\frac{2^{-2(\beta_\mu-1)}}{12}. \tag{B1}$$

Combining (10) and (11) with (6), we get a recursive expression for the accumulated numerical error in the computation of $\sigma_e(n|m)$ as

$$\begin{aligned}\epsilon_\sigma(n|m) &= \beta\epsilon_\sigma(n-1|m) + 2(1-\beta)\\ &\quad\cdot\{e_f(n|m-1)\,\epsilon_f(n|m-1)\\ &\quad+ e_b(n-1|m-1)\,\epsilon_b(n-1|m-1)\}\\ &\quad+(1-\beta)\{\epsilon_f^2(n|m-1)\\ &\quad+\epsilon_b^2(n-1|m-1)\} + \eta_\sigma(n|m).\end{aligned} \tag{B2}$$

Squaring (B2), taking the statistical expectations (as n goes to ∞) of both sides, and simplifying by neglecting all error terms of order three or more, and also all the products of uncorrelated error terms, we get the following expression for $E\{\epsilon_\sigma^2(m)\}$:

$$E\{\epsilon_\sigma^2(m)\} = \frac{\sigma_\eta^2 + 8(1-\beta)^2 E\{e_f^2(m-1)\}\,E\{\epsilon_f^2(m-1)\}}{(1-\beta^2)}. \tag{B3}$$

It is straightforward to show that in the steady-state

$$E\{\epsilon_\sigma(m)\} = 2E\{\epsilon_f^2(m-1)\}, \tag{B4}$$

and

$$E\{\sigma_e(m)\} = 2E\{e_f^2(m-1)\}. \tag{B5}$$

Then using an approximate result similar to that in (19),

$$E\left\{\frac{\mu}{(\sigma_e(m))^l}\right\} \approx \frac{\mu}{\left[2\{E\{e_f^2(m-1)\} + E\{\epsilon_f^2(m-1)\}\}\right]^l}. \tag{B6}$$

Now, let us consider the normalized, finite precision update equation for the reflection coefficients. We will assume that the order in which the update is done is: evaluation of $e_f'(n|m)$, $e_b'(n-1|m-1) + e_b'(n|m)\,e_f'(n|m-1)$ (which will produce a roundoff error with variance $2\cdot 2^{-2(b_c-1)}/12$), division of this quantity by $\sigma_e'(n|m)$, and then the multiplication by μ. Then the variance of $\eta_{kN}(n|m)$ is given by (in the steady-state)

$$E\{\eta_{kN}^2(m)\} \approx \frac{2^{-2(b_c-1)}}{12}\left\{1+\mu^2 \cdot\left(1+\frac{2}{\{2(E\{e_f^2(m-1)\}+E\{\epsilon_f^2(m-1)\})\}^2}\right)\right\}. \tag{B7}$$

Now we can square both sides of (14), take the statistical expectations as n goes to ∞, and simplify using several of the assumptions and approximations used earlier to get (the steady-state result)

$$\begin{aligned}E\{\xi^2(m)\} &= E\{\eta_{kN}^2(m)\} + \frac{\mu^2 E\{\epsilon_\sigma^2(m)\}\,E\{e_f^2(m-1)\}}{4E^4\{e_f^2(m-1)\}}\\ &\quad\cdot\{1 - 4k_{\text{opt}}^2(m) + E\{k_m^2\}\,(2+k_{\text{opt}}^2(m))\}.\end{aligned} \tag{B8}$$

In order derive (B8), we made use of the fact that the steady-state, infinite precision prediction error sequences have zero mean values and are Gaussian, and therefore simplified the fourth-order expectations using the result in (A4). Furthermore, we made use of the approximation in (A5).

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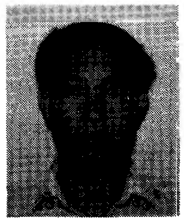

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